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Fermion Tunneling from an Axis-Symmetric Black Hole beyond the Semiclassical Approximation

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Abstract Based on the Hamilton-Jacobi method beyond the semiclassical approximation proposed by R. Baberjee and B.R. Majhi, Hawking radiation of Dirac particle as tunneling through the event horizon is calculated. It is shown that all quantum corrections in the fermion particle action are proportional to the usual semiclassical contribution. Under the conception of irreducible mass and the first law of thermodynamics, the modifications to Hawking temperature and Bekenstein-Hawking entropy are given for a Kerr-Newmann black hole.

Keywords Beyond the semiclassical approximation · Axis-symmetric black hole · Black hole entropy correction · Hamilton-Jacobi method

1 Introduction

Since Hawking [1] startled the physics community by proving that black holes are not black and they radiate energy continuously, semiclassical methods of modeling Hawking radiation as a tunneling effect were developed and have gained a lot of interest [1–27]. The essence of tunneling based calculation is the computation of the imaginary part of the action for the process of s-wave emission across the horizon, which in turn is related to the Boltzmann factor for the emission at Hawking temperature. There are two different methods to calculate the imaginary part of the particle's action: One is Parikh-Wilzeck's radial null geodesic method [4] and the other is Hamilton-Jacobi method which was first used by Srinivasan et al. [5]. After that, many people applied these two methods to find out Hawking temperature for more general space-times [10–12]. In the meantime, the tunneling of Dirac particle was also investigated [15–18].

However, all these research was confined to the semiclassical approximation. In [28], R. Baberjee and B.R. Majhi firstly proposed the Hamilton-Jacobi method beyond semiclassical approximation. They considered all terms in the expansion of the action for a scalar

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particle. The result showed that the higher order terms are proportional to the semiclassical contribution. Due to dimensional argument, the form of these proportionality constants are determined in order to get dimensionless parameters. This result also, together with properties of conformal transformation, eventually led to corrected expressions for thermodynamic variables of a black hole [29]. A similar analysis for the case of fermion tunneling from a stationary black hole is also studied [30], and then the tunneling from a Friedmann-Robertson-Walker universe was discussed in [31]. They are all spherical spacetimes. However, it is not known whether a similar analysis is valid for the case of fermion tunneling from an axis-symmetric black hole. Using the irreducible mass instead of the mass of an axis-symmetric black hole, it is shown that the higher order terms in the single particle action are also proportional to the semiclassical contribution. Thinking of the tunneling probability and radiation spectrum, the correction to the Bekenstein-Hawking area law is derived under the first law of black hole thermodynamics. Interestingly, the leading order correction to the entropy is the logarithmic of the semiclassical entropy which is consistent with [28, 30].

In Sect. 2, for a general axis-symmetric spacetime, we will calculate the Dirac particle tunneling beyond semiclassical approximation via the Hamilton-Jacobi method suggested in [28]. In Sect. 3, the results for a Kerr-Newmann black hole are discussed and given. In Sect. 4, some discussions and conclusions are given.

2 Hamilton-Jacobi Method beyond the Semiclassical Approximation

A general axis-symmetric space-time can be expressed

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{\theta\theta}d\theta^2. \tag{1}$$

The massless Dirac equation in a curved spacetime is given by

$$i\gamma^\mu \nabla_\mu \Psi = 0, \tag{2}$$

where the covariant derivative is given by $\nabla_\mu = \partial_\mu + \frac{i}{2}\Gamma_\mu^{\alpha\beta}\Sigma_{\alpha\beta}$, and $\Gamma_\mu^{\alpha\beta} = g^{\beta\nu}\Gamma^\alpha_{\mu\nu}$, $\Sigma_{\alpha\beta} = \frac{i}{4}[\gamma_\alpha, \gamma_\beta]$.

The gamma matrices in a curved space-time is

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}I. \tag{3}$$

After the general gamma matrices are defined, we prefer to choose the axis-symmetric gamma matrices as

$$\begin{aligned} &\gamma^t = \sqrt{-g^{tt}}\gamma^0, \qquad \gamma^r = \sqrt{g^{rr}}\gamma^3, \qquad \gamma^\theta = \sqrt{g^{\theta\theta}}\gamma^1, \\ &\gamma^\phi = \sqrt{g^{\phi\phi} + \frac{(g^{t\phi})^2}{-g^{tt}}}\gamma^2 + \sqrt{\frac{(g^{t\phi})^2}{-g^{tt}}}\gamma^0, \end{aligned} \tag{4}$$

where $\gamma^0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}$, $\gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}$, $\gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}$, and the Pauli Matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To solve (2), we employ the following ansatz for spin up (i.e. positive r direction) and spin down (i.e. negative r direction) Ψ as

$$\Psi_\uparrow = \begin{pmatrix} A(t,r,\theta,\varphi) \\ 0 \\ B(t,r,\theta,\varphi) \\ 0 \end{pmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow(t,r,\theta,\varphi)\right], \tag{5}$$

$$\Psi_\downarrow = \begin{pmatrix} C(t,r,\theta,\varphi) \\ 0 \\ D(t,r,\theta,\varphi) \\ 0 \end{pmatrix} \exp\left[\frac{i}{\hbar} I_\downarrow(t,r,\theta,\varphi)\right], \tag{6}$$

where $I_\uparrow$ (i.e. $I_\downarrow$) is the particle's action which will be expanded in power of $\hbar$.

In the following we will only solve the spin up case explicitly since the spin down case is analogous. In order to apply the method beyond semiclassical approximation, we can insert the ansatz I for a spin field $\Psi(t,r,\theta,\varphi)$ into the general covariant Dirac equation. Putting (4), (5) into (2), dividing by the exponential term and neglecting the terms including $\hbar$, we have

$$iA\left(\sqrt{-g^{tt}}\partial_t I + \frac{-g^{t\phi}}{\sqrt{-g^{tt}}}\partial_\phi I\right) + B\sqrt{g^{rr}}\partial_r I = 0, \tag{7}$$

$$\left(\sqrt{g^{\theta\theta}}\partial_\theta I + i\sqrt{g^{\phi\phi} + \frac{(g^{t\phi})^2}{-g^{tt}}}\partial_\phi I\right)B = 0, \tag{8}$$

$$-iB\left(\sqrt{-g^{tt}}\partial_t I + \frac{-g^{t\phi}}{\sqrt{-g^{tt}}}\partial_\phi I\right) + A\sqrt{g^{rr}}\partial_r I = 0, \tag{9}$$

$$\left(\sqrt{g^{\theta\theta}}\partial_\theta I + i\sqrt{g^{\phi\phi} + \frac{(g^{t\phi})^2}{-g^{tt}}}\partial_\phi I\right)A = 0. \tag{10}$$

Neglecting the term including $\hbar$ is because those terms do not involve the fermion particle's action and they do not contribute the thermodynamics entities of the black hole. From (8) and (10), we can get

$$\sqrt{g^{\theta\theta}}\partial_\theta I + i\sqrt{g^{\phi\phi} + \frac{(g^{t\phi})^2}{-g^{tt}}}\partial_\phi I = 0. \tag{11}$$

Now expanding I, A, and B in power of $\hbar$ as

$$I(t,r,\theta,\varphi) = I_0(t,r,\theta,\varphi) + \sum_i \hbar^i I_i(t,r,\theta,\varphi), \tag{12}$$

$$A(t,r,\theta,\varphi) = A_0(t,r,\theta,\varphi) + \sum_i \hbar^i A_i(t,r,\theta,\varphi), \tag{13}$$

$$B(t,r,\theta,\varphi) = B_0(t,r,\theta,\varphi) + \sum_i \hbar^i B_i(t,r,\theta,\varphi), \tag{14}$$

where $i = 1, 2, 3, \ldots$. In these expansions, the terms from $O(\hbar)$ onward are treated as quantum corrections over the semiclassical value $I_0(t,r,\theta,\varphi)$, $A_0(t,r,\theta,\varphi)$, and $B_0(t,r,\theta,\varphi)$

respectively. Substituting those equations into (7) and (9), then equating the different powers of $\hbar$ on both sides, we obtain the following two sets of equations

$$\hbar^0 : i A_0\left(\sqrt{-g^{tt}}\partial_t I_0 + \frac{-g^{t\phi}}{\sqrt{-g^{tt}}}\partial_\phi I_0\right) + B_0\sqrt{g^{rr}}\partial_r I_0 = 0, \tag{15}$$

$$\begin{aligned}\hbar^1 :\ & i\sqrt{-g^{tt}}(A_1\partial_t I_0 + A_0\partial_t I_1) - i\frac{g^{t\phi}}{\sqrt{-g^{tt}}}(A_1\partial_\phi I_0 + A_0\partial_\phi I_1)\\ & + \sqrt{g^{rr}}(B_1\partial_r I_0 + B_0\partial_r I_1) = 0,\end{aligned} \tag{16}$$

$$\begin{aligned}\hbar^2 :\ & i\sqrt{-g^{tt}}(A_2\partial_t I_0 + A_1\partial_t I_1 + A_0\partial_t I_2) - i\frac{g^{t\phi}}{\sqrt{-g^{tt}}}(A_2\partial_\phi I_0 + A_1\partial_\phi I_1 + A_0\partial_\phi I_2)\\ & + \sqrt{g^{rr}}(B_2\partial_\phi I_0 + B_1\partial_\phi I_1 + B_0\partial_r I_2) = 0,\end{aligned} \tag{17}$$

$$\vdots$$

$$\hbar^0 : -i B_0\left(\sqrt{-g^{tt}}\partial_t I_0 + \frac{-g^{t\phi}}{\sqrt{-g^{tt}}}\partial_\phi I_0\right) + A_0\sqrt{g^{rr}}\partial_r I_0 = 0, \tag{18}$$

$$\begin{aligned}\hbar^1 :\ & -i\sqrt{-g^{tt}}(B_1\partial_t I_0 + B_0\partial_t I_1) + i\frac{g^{t\phi}}{\sqrt{-g^{tt}}}(B_1\partial_\phi I_0 + B_0\partial_\phi I_1)\\ & + \sqrt{g^{rr}}(A_1\partial_r I_0 + A_0\partial_r I_1) = 0,\end{aligned} \tag{19}$$

$$\begin{aligned}\hbar^2 :\ & -i\sqrt{-g^{tt}}(B_2\partial_t I_0 + B_1\partial_t I_1 + B_0\partial_t I_2) + i\frac{g^{t\phi}}{\sqrt{-g^{tt}}}(B_2\partial_\phi I_0 + B_1\partial_\phi I_1 + B_0\partial_\phi I_2)\\ & + \sqrt{g^{rr}}(A_2\partial_\phi I_0 + A_1\partial_\phi I_1 + A_0\partial_r I_2) = 0.\end{aligned} \tag{20}$$

$$\vdots$$

It is evident that the first equations in the upper two sets (15) and (18) are the semiclassical Hamilton-Jacobi equations for a Dirac particle. These two equations have a non-trivial solution for A_0 and B_0 if and only if the determinant of the coefficient matrix vanishes

$$\begin{vmatrix} i\left(\sqrt{-g^{tt}}\partial_t I_0 + \frac{-g^{t\phi}}{\sqrt{-g^{tt}}}\partial_\phi I_0\right) & \sqrt{g^{rr}}\partial_r I_0 \\ \sqrt{g^{rr}}\partial_r I_0 & -i\left(\sqrt{-g^{tt}}\partial_t I_0 + \frac{-g^{t\phi}}{\sqrt{-g^{tt}}}\partial_\phi I_0\right) \end{vmatrix} = 0, \tag{21}$$

then we can get

$$\left(\sqrt{-g^{tt}}\partial_t I_0 + \frac{-g^{t\phi}}{\sqrt{-g^{tt}}}\partial_\phi I_0\right)^2 - g^{rr}(\partial_r I_0)^2 = 0. \tag{22}$$

Because there are two Killing vectors $(\partial/\partial t)^\mu$ and $(\partial/\partial\phi)^\mu$ in an axis-symmetric space-time, we can separate the variables for $I_0(t, r, \theta, \varphi)$ as follows

$$I_0 = -\omega t + j\phi + R(r,\theta) + K, \tag{23}$$

where ω and j are the Dirac particle's energy and angular momentum respectively, and K is a complex constant. Inserting (23) into (22), one can arrive at

$$\left(\sqrt{-g^{tt}}\omega + \frac{-g^{t\phi}}{\sqrt{-g^{tt}}} j\right)^2 - g^{rr}(\partial_r R)^2 = 0, \tag{24}$$

it indicates that $R(r,\theta)$ is a complex function. Now the equation has two possible solutions for definite θ_0 as [15, 32, 33]

$$R_+(r,\theta_0) = \int dr \frac{1}{\sqrt{g^{rr}}}\left(\sqrt{-g^{tt}}\omega - \frac{g^{t\phi}}{\sqrt{-g^{tt}}} j\right), \quad \text{while } A_0 = iB_0, \tag{25}$$

$$R_-(r,\theta_0) = -\int dr \frac{1}{\sqrt{g^{rr}}}\left(\sqrt{-g^{tt}}\omega - \frac{g^{t\phi}}{\sqrt{-g^{tt}}} j\right), \quad \text{while } A_0 = -iB_0, \tag{26}$$

where $R_+(r,\theta_0)(R_-(r,\theta_0))$ corresponds to the ingoing(outgoing) solution.

Now it is interesting to get the relations between different order in the expansion of A and B as

$$A_\alpha = \pm i B_\alpha, \tag{27}$$

where $\alpha = 0, 1, 2, 3, \ldots$. This will lead to a simplified form of all the equations from (15) to (20) as

$$\left(\sqrt{-g^{tt}}\partial_t I_\alpha + \frac{-g^{t\phi}}{\sqrt{-g^{tt}}}\partial_\phi I_\alpha\right)^2 - g^{rr}(\partial_r I_\alpha)^2 = 0, \tag{28}$$

so the functional form of the above individual linear differential equations is the same and is identical to the usual semiclassical Hamilton-Jacobi equation. Obviously, the solutions of these equations are not independent and I_α's are proportional to I_0. This result is consistent with [28–30]. A similar situation happened for scalar particle tunneling and fermion particle tunneling from the spherical black holes.

Considering of (12), the proportionality constants of I_i should have the dimension of $\hbar^{-i}$ in order to the dimensions of I_i and I_0 should be same. In the units $G = c = k_B = 1$, the Planck constant $\hbar$ is the order of squared Planck mass M_p, and so from dimensional analysis the proportionality constants have the dimension of M_{ir}^{-2i} where M_{ir} is the irreducible mass of an axis-symmetric black hole. We have chosen M_{ir} instead of M because M_{ir} is an invariant in a reversible process [34]. For a rotating black hole in a reversible process, its irreducible mass M_{ir} is unchanged [35, 36]. Therefore, the introduction of the irreducible mass ensures that the corrections to B-H entropy is unchanged in a reversible process. For definite θ_0, thinking of (22) and (28), the most general expression for I, can be rewritten as

$$I(t,r,\theta_0,\varphi) = \left(1 + \sum_i \beta_i \frac{\hbar^i}{M_{ir}^{2i}}\right) I_0(t,r,\theta_0,\varphi), \tag{29}$$

where β_i's are some dimensionless constant parameters.

The above analysis shows that to obtain a solution for $I(t,r,\theta_0,\varphi)$, it is enough to solve the equation of $I_0(t,r,\theta_0,\varphi)$ which has the solution form as (23). In fact the standard Hamilton-Jacobi solution determined by this $I_0(t,r,\theta_0,\varphi)$ is just modified by a prefactor to yield the complete solution for $I(t,r,\theta_0,\varphi)$. Substituting (23) into (29), we obtain

$$I(t,r,\theta_0,\varphi) = \left(1 + \sum_i \beta_i \frac{\hbar^i}{M_{ir}^{2i}}\right)[-\omega t + R_\pm(r,\theta_0) + j\phi + K], \tag{30}$$

so the ingoing and outgoing solutions of the Dirac equation (2) under the background metric equation (1) is given by exploiting (22) and (28)

$$\Psi_{in} \sim \exp\left[-\frac{i}{\hbar}\left(1+\sum_i \beta_i \frac{\hbar^i}{M_{ir}^{2i}}\right)(R_-(r,\theta_0)+K)\right], \tag{31}$$

$$\Psi_{out} \sim \exp\left[-\frac{i}{\hbar}\left(1+\sum_i \beta_i \frac{\hbar^i}{M_{ir}^{2i}}\right)(R_+(r,\theta_0)+K)\right]. \tag{32}$$

Now just as discussed in [37, 38], one solution corresponds to the Dirac particles moving away from the event horizon and the other solution corresponds to the particles incoming toward the event horizon. The probabilities of crossing the outer horizon are, respectively, given by

$$P_{out} \sim \exp\left[-\frac{2}{\hbar}\left(1+\sum_i \beta_i \frac{\hbar^i}{M_{ir}^{2i}}\right)(\operatorname{Im} R_+(r,\theta_0)+\operatorname{Im} K)\right], \tag{33}$$

$$P_{in} \sim \exp\left[-\frac{2}{\hbar}\left(1+\sum_i \beta_i \frac{\hbar^i}{M_{ir}^{2i}}\right)(\operatorname{Im} R_-(r,\theta_0)+\operatorname{Im} K)\right]. \tag{34}$$

To ensure that the probability is normalized, we should note that the probability of any incoming wave crossing the outer horizon is unity [38], so we have $\operatorname{Im} R_+ = -\operatorname{Im} R_-$. This implies that the probability of a Dirac particle tunneling from inside to outside the event horizon should be

$$\begin{aligned}\Gamma &= \exp\left[-\frac{4}{\hbar}\left(1+\sum_i \beta_i \frac{\hbar^i}{M_{ir}^{2i}}\right)\operatorname{Im} R_+(r,\theta_0)\right]\\ &= \exp\left[-\frac{4}{\hbar}\left(1+\sum_i \beta_i \frac{\hbar^i}{M_{ir}^{2i}}\right)\operatorname{Im}\int dr\frac{1}{\sqrt{g^{rr}}}\left(\sqrt{-g^{tt}}\omega-\frac{g^{t\phi}}{\sqrt{-g^{tt}}}j\right)\right].\end{aligned} \tag{35}$$

Using the expression under the metric coefficients g^{rr}, g^{tt}, and $g^{t\phi}$ are given, one can easily find out the temperature of the corresponding axis-symmetric black hole. The same result has been obtained in [27] for scalar particle tunneling and in [30] for Dirac particle tunneling in a general class of static spherically symmetric spacetime.

3 Tunneling from a Kerr_Newmann Black Hole

Now we will consider the Kerr-Newmann metric to show how the semiclassical Hawking temperature can be given. The correction to the Bekenstein entropy is also calculated under the first thermodynamics law and the irreducible mass of the black hole.

For a Kerr-Newmann black hole, the metric is

$$ds^2 = -f(r,\theta)dt^2 + \frac{dr^2}{g(r,\theta)} - 2H(r,\theta)dtd\phi + K(r,\theta)d\phi^2 + \Sigma(r,\theta)d\theta^2, \tag{36}$$

where $f(r,\theta) = \frac{\triangle(r)-a^2\sin^2\theta}{\Sigma(r,\theta)}$, $g(r,\theta) = \frac{\triangle(r)}{\Sigma(r,\theta)}$, $H(r,\theta) = \frac{a^2\sin^2\theta(r^2+a^2-\triangle(r))}{\Sigma(r,\theta)}$, $K(r,\theta) = \frac{(r^2+a^2)^2-\triangle(r)a^2\sin^2\theta}{\Sigma(r,\theta)}\sin^2\theta$, $\Sigma(r,\theta) = r^2+a^2\cos^2\theta$, and $\triangle(r) = r^2+a^2+e^2-2Mr$.

It is easy to get

$$g^{rr} = g(r,\theta) = \frac{\triangle(r)}{\Sigma(r,\theta)}, \qquad g^{tt} = \frac{K(r,\theta)}{f(r,\theta)K(r,\theta)+H^2(r,\theta)},$$
$$g^{t\phi} = -\frac{H(r,\theta)}{f(r,\theta)K(r,\theta)+H^2(r,\theta)}.$$

Considering the Dirac particle tunneling through the out horizon at fixed $\theta = \theta_0$, the final result for $R_+(r_+,\theta_0)$ is given [32]

$$R_+(r_+,\theta_0) = (\omega - \Omega_H j)\frac{2i\pi(r_+^2+a^2)}{(r_+-r_-)}, \tag{37}$$

where $\Omega_H = \frac{H(r_+,\theta_0)}{K(r_+,\theta_0)} = \frac{a}{r_+^2+a^2}$. Using (35) and (37), the probability of a Dirac particle tunneling from inside to outside of the event horizon is given by

$$\Gamma = \exp\left[-\frac{4}{\hbar}\left(1+\sum_i \beta_i \frac{\hbar^i}{M_{ir}^{2i}}\right)(\omega - \Omega_H j)\frac{2\pi(r_+^2+a^2)}{(r_+-r_-)}\right]. \tag{38}$$

From the tunneling probability above, the fermion spectrum of Hawking radiation from a Kerr-Newmann black hole can be deduced [39, 40]

$$N(\omega, j) = \frac{1}{e^{2\pi(\omega-\Omega_H j)/\kappa}+1}, \tag{39}$$

where $\kappa = \frac{r_+-r_-}{2(r_+^2+a^2)}(1+\sum_i \beta_i \frac{\hbar^i}{M_{ir}^{2i}})^{-1}$ is the surface gravity of the event horizon. From the tunneling probability and radiation spectrum, Hawking temperature of a Kerr-Newmann black hole can be determined as

$$T_h = \frac{\kappa}{2\pi} = \frac{r_+-r_-}{4\pi(r_+^2+a^2)}\left(1+\sum_i \beta_i \frac{\hbar^i}{M_{ir}^{2i}}\right)^{-1} = T_H\left(1+\sum_i \beta_i \frac{\hbar^i}{M_{ir}^{2i}}\right)^{-1}, \tag{40}$$

where $T_H = \frac{r_+-r_-}{4\pi(r_+^2+a^2)}$ is the standard semiclassical Hawking temperature of the black hole and other terms are the corrections due to the quantum effect.

Thinking of Gibbs form of the first law of thermodynamics, the corrected form of the Bekenstein-Hawking entropy can be given. Using the irreducible mass, we have

$$M_{ir}^2 = \frac{A}{16\pi},$$

where $A = 4\pi(r_+^2+a^2) = 4\pi[2M^2 - Q^2 + 2(M^4 - J^2 - M^2Q^2)^{1/2}]$, which is the area of the event horizon of the Kerr-Newmann black hole. According to the equation above, we obtain

$$2M_{ir}dM_{ir} = \frac{r_+^2+a^2}{r_+-r_-}(dM - VdQ - \Omega dJ), \tag{41}$$

where $V = \frac{Qr_+}{r_+^2+a^2}$, $\Omega = \frac{a}{r_+^2+a^2}$, and $a = J/M$. The first law of thermodynamics is given as

$$T_h dS_{bh} = (dM - VdQ - \Omega dJ), \tag{42}$$

so we get

$$2M_{ir}dM_{ir} = \frac{r_+^2 + a^2}{r_+ - r_-} T_h dS_{bh}. \tag{43}$$

To simplify the equation above, we obtain

$$dM_{ir} = \frac{1}{8\pi M_{ir}} \left(1 + \sum_i \beta_i \frac{\hbar^i}{M_{ir}^{2i}}\right)^{-1} dS_{bh},$$

so we have

$$\begin{aligned} S_{bh} &= \int 8\pi M_{ir} \left(1 + \sum_i \beta_i \frac{\hbar^i}{M_{ir}^{2i}}\right) dM_{ir} \\ &= 4\pi M_{ir}^2 + 8\pi \beta_1 \hbar \ln M_{ir} - \frac{4\pi \hbar^2 \beta_2}{M_{ir}^2} + \text{higher order terms in } \hbar. \end{aligned} \tag{44}$$

Using the area of the event horizon of the Kerr-Newmann black hole, we have

$$S_{bh} = \frac{A}{4} + 4\pi \beta_1 \hbar \ln A - \frac{64\pi^2 \hbar^2 \beta_2}{A} + \cdots. \tag{45}$$

It is obvious that the first term is the usual semiclassical contribution to the area law $S_{BH} = \frac{A}{4}$. The other terms are the quantum corrections. Now it is possible to express the quantum corrections in terms of S_{BH} instead of A as

$$S_{bh} = S_{BH} + 4\pi \beta_1 \ln S_{BH} - \frac{16\pi^2 \beta_2}{S_{BH}} + \cdots.$$

Interestingly the leading order correction is logarithmic in A or S_{BH} which was found earlier in [41, 42] by field theory calculations and later in [43, 44] by the quantum geometry method. The higher order corrections involve inverse powers of A or S_{BH}. The coefficient of the logarithmic term of entropy and other terms are possibly related to trace anomaly.

4 Conclusions

We have successfully extended the approach of scalar particle tunneling beyond semiclassical approximation [28] to fermion tunneling from an axis-symmetric black hole. Considering all orders in the single particle action of fermion tunneling through the event horizon of the axis-symmetric black hole, we found that higher order correction terms of the action are proportional to the semiclassical contribution. A similar result was shown earlier in [28] for the scalar particle tunneling. By dimensional argument and principle of "detailed balance", the same form of the modified Hawking temperature, as in the scalar case, was given. The logarithmic and inverse powers of area corrections to the Bekenstein-Hawking area law were also given using the irreducible mass of the axis-symmetric black hole.

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